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Sufficiency and Duality in Multiobjective Variational Problems with Generalized Type I Functions

MOHAMED HACHIMI[1] and BRAHIM AGHEZZAF[2]
[1]*Faculté des Sciences Juriduques Economiques et Sociales, Université Ibn Zohr, B.P. 8658 Hay Dakhla, Agadir 80 000 Morocco (e-mail: hachimi@esta.ac.ma)*
[2]*Département de Mathématiques et d'Informatique Faculté des Sciences, Université Hassan II-Aïn chock, B.P. 5366 - Maârif Casablanca Morocco (e-mail: b.aghezzaf@fsac.ac.ma)*

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Abstract. Recently Hachimi and Aghezzaf introduced the notion of (F, α, ρ, d)-type I functions, a new class of functions that unifies several concepts of generalized type I functions. Here, we extend the concepts of (F, α, ρ, d)-type I and generalized (F, α, ρ, d)-type I functions to the continuous case and we use these concepts to establish various sufficient optimality conditions and mixed duality results for multiobjective variational problems. Our results apparently generalize a fairly large number of sufficient optimality conditions and duality results previously obtained for multiobjective variational problems.

Key words: Multiobjective variational programming, Optimality, Duality, Efficient Solution, Properly efficient solution, Generalized (F, α, ρ, d)-type I functions

1. Introduction

Investigation on sufficiency and/or duality has been one of the most attracting topics in the theory of multiobjective problems. In multiobjective (static) programming problems, convexity plays an important role in deriving sufficient optimality conditions and duality results. Several classes of functions have been defined for the purpose of weakening the limitations of convexity in nonlinear programming problems. The concept of type I functions was first introduced by Hanson and Mond [12] as a generalization of convexity. Subsequently, Rueda and Hanson [22] have defined pseudo-type I and quasi-type I functions and have obtained sufficient optimality conditions involving these functions. Kaul et al. [13] obtained optimality conditions and duality results for multiobjective programming problems involving type I and generalized type I functions. Later, Aghezzaf and Hachimi [1] have introduced generalized type I functions, for multiobjective programming problems, which are different from those defined in Kaul et al. [13] and have obtained some duality results. In recent paper, Hachimi and Aghezzaf [10] have defined generalized (F, α, ρ, d)-type I functions, a new class of functions that unifies several concepts of generalized type I functions. They have obtained sufficient optimality conditions and duality for multiobjective programming problems.

In this paper, we extend the concepts of (F, α, ρ, d)-type I and generalized (F, α, ρ, d)-type I functions to the continuous case and we use these concepts to establish sufficient optimality conditions and mixed duality results for multiobjective variational programming problems. The results obtained in this paper are more general than those obtained in the references [1, 5, 10, 13, 20, 17, 19, 24].

2. Notations and statement of the problem

Let $I=[a,b]$ be a real interval and $f=(f^1,\dots,f^p): I\times\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^p$ and $g=(g^1,\dots,g^q): I\times\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^q$ be continuously differentiable functions. In order to consider $f(t,x,\dot{x})$, where $x: I\longrightarrow\mathbb{R}^n$ is differentiable with derivative $\dot{x}$, denote the $p\times n$ matrices of first partial derivatives of f with respect to $x, \dot{x}$ by f_x and $f_{\dot{x}}$, such that

$$f_x^i=\left(\frac{\partial f^i}{\partial x_1},\dots,\frac{\partial f^i}{\partial x_n}\right) \text{ and } f_{\dot{x}}^i=\left(\frac{\partial f^i}{\partial \dot{x}_1},\dots,\frac{\partial f^i}{\partial \dot{x}_n}\right), \quad i=1,2,\dots,p.$$

Similarly, g_x and $g_{\dot{x}}$ denote the $m\times n$ matrices of first partial derivatives of g with respect to x and $\dot{x}$. Let $\mathscr{C}(I,\mathbb{R}^n)$ denote the space of continuously differentiable functions x with norm $\|x\|:=\|x\|_\infty+\|\mathrm{D}x\|_\infty$, where the differential operator D is given by

$$u=\mathrm{D}x \Longleftrightarrow x(t)=x(a)+\int_a^t u(s)\mathrm{d}s.$$

Therefore, $\mathrm{D}=\mathrm{d}/\mathrm{d}t$ except at discontinuities. We consider the following multiobjective variational problem,

$$\text{(MOP)} \quad \text{Minimize} \quad \int_a^b f(t,x,\dot{x})\,\mathrm{d}t$$

$$=\left(\int_a^b f^1(t,x,\dot{x})\,\mathrm{d}t,\dots,\int_a^b f^p(t,x,\dot{x})\,\mathrm{d}t\right)$$

$$\text{subject to} \qquad x(a)=\alpha, \quad x(b)=\beta, \tag{1a}$$

$$g(t,x,\dot{x})\leqq 0, \quad t\in I. \tag{1b}$$

Let $A=\{x\in\mathscr{C}(I,\mathbb{R}^n), x(a)=\alpha, x(b)=\beta, g(t,x,\dot{x})\leqq 0, \forall t\in I\}$ be the set of feasible solutions for (MOP).

Notations Throughout this paper we use the following notations. Let $P=\{1,2,\dots,p\}$ and $Q=\{1,2,\dots,q\}$ be the index sets, and let $\{J_1, J_2\}$ be a partition of Q. For a vector v and vector-valued function g, v_{J_1} and g^{J_1} denote the subvectors composed from the components $v_i, i \in J_1$ and the subvector-valued function composed from the components $g^i, i \in J_1$, respectively. For a vector $\bar{u} \in \mathbb{R}^p$ and function $f = (f^1, \ldots, f^p), \overline{U} f$ denotes the function $(\overline{u}_1 f^1, \ldots, \overline{u}_p f^p)$.

If x and $y \in \mathbb{R}^n$, then $x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, \ldots, n$; $x \leqslant y \Leftrightarrow x \leqq y$ and $x \neq y$; $x < y \Leftrightarrow x_i < y_i, i = 1, \ldots, n$; xy or $x^t y$ denotes the inner product. Let e be the vector of $\mathbb{R}^p$ whose components are all ones. For $\rho \in \mathbb{R}^p$ and $\alpha \in \mathbb{R}$, $e\rho$ and αe denote the scalar $\sum_1^p \rho_i$ and the vector whose components are all α, respectively.

For the multiobjective variational programming problem (MOP), the solution is defined in terms of proper efficient and (weak) efficient solution in the following sense [4]:

DEFINITION 2.1. A point $\bar{x} \in A$ is said to be an efficient solution for problem (MOP) if there exists no other $x \in A$ such that

$$\int_a^b f(t, x, \dot{x}) \mathrm{d}t \leqslant \int_a^b f(t, \bar{x}, \dot{\bar{x}}) \mathrm{d}t. \tag{2}$$

DEFINITION 2.2. A point $\bar{x} \in A$ is said to be a weak efficient solution for problem (MOP) if there exists no other $x \in A$ such that

$$\int_a^b f(t, x, \dot{x}) \mathrm{d}t < \int_a^b f(t, \bar{x}, \dot{\bar{x}}) \mathrm{d}t. \tag{3}$$

DEFINITION 2.3. A point $\bar{x} \in A$ is said to be a properly efficient solution for problem (MOP) if there exists a scalar $M > 0$ such that, $\forall i \in P$,

$$\int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) \mathrm{d}t - \int_a^b f^i(t, x, \dot{x}) \mathrm{d}t \leqq M \left(\int_a^b f^j(t, x, \dot{x})\, \mathrm{d}t - \int_a^b f^j(t, \bar{x}, \dot{\bar{x}}) \mathrm{d}t \right) \tag{4a}$$

for some j such that, for x in A,

$$\int_a^b f^j(t, x, \dot{x}) \mathrm{d}t > \int_a^b f^j(t, \bar{x}, \dot{\bar{x}}) \mathrm{d}t \quad \text{and}$$
$$\int_a^b f^i(t, x, \dot{x}) \mathrm{d}t < \int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) \mathrm{d}t. \tag{4b}$$

In the case of maximization, the signs of inequalities (2), (3), (4a) and (4b) are reversed (*i.e.* we replace the signs $\leqslant, <$ and $>$ by $\geqslant, >$ and $<$, respectively).

3. Generalized (F, α, ρ, d)-type I functions

In this section, we will extend the concepts of (F, α, ρ, d)-type I and generalized (F, α, ρ, d)-type I functions defined for multiobjective static programming problems in [10] to the multi objective variational programming problems. The following definition of sublinear functional was given in [18]:

DEFINITION 3.1. A functional $F : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is sublinear with respect to the sixth variable if for any $x, \bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$,

$$F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}; a_1 + a_2) \leqq F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}; a_1) + F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}; a_1) \quad \forall a_1, a_2 \in \mathbb{R}^n; \tag{5a}$$

$$F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}; \lambda a) = \lambda F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}; a) \quad \forall \lambda \in \mathbb{R},\ \lambda \geqq 0, \quad \forall a \in \mathbb{R}^n. \tag{5b}$$

From (5b) it follows $F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}; 0) = 0$.

Let F be a sublinear functional and the functions $(f = f^1, \dots, f^p)$: $I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $h = (h^1, \dots, h^r) : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^r$ be continuously differentiable with respect to each of their arguments. Let $\rho = (\rho^1, \rho^2)$ where $\rho^1 = (\rho_1, \dots, \rho_p) \in \mathbb{R}^p$, $\rho^2 = (\rho_{1+p}, \dots, \rho_{r+p}) \in \mathbb{R}^r$. Let $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^1, \alpha^2 : \mathscr{C}(I, \mathbb{R}^n) \times \mathscr{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}_+ \backslash \{0\}$, and let $d : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

For the sake of simplicity, we will use the following notation. If F is a sublinear functional, $\psi : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\alpha^i : \mathscr{C}(I, \mathbb{R}^n) \times \mathscr{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}_+ \backslash \{0\}$, then

$$\mathscr{F}(\psi, t, x, y, \alpha^i) = F\Big(t, x, \dot{x}, y, \dot{y}; \alpha^i(x, y)\Big[\psi_x(t, y, \dot{y}) - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{x}}(t, y, \dot{y})\Big]\Big) \tag{6}$$

We note that $\mathscr{F}(\psi, t, x, y, \alpha^i) = \alpha^i(x, y)\mathscr{F}(\psi, t, x, y, \mathbf{1})$ where $\mathbf{1} : \mathscr{C}(I, \mathbb{R}^n) \times \mathscr{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}$ such that for every $x, y \in \mathbb{R}^n$, $\mathbf{1}(x, y) = 1$.

If $f : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$, then the symbol $\mathscr{F}(f, t, x, y, \alpha^i)$ denotes the vector of components $\mathscr{F}(f^1, t, x, y, \alpha^i), \dots, \mathscr{F}(f^p, t, x, y, \alpha^i)$.

The following remark will be used in the sequel.

Remark 3.1. Let $f : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$ *be continuously differentiable function and* $\alpha^1 : \mathscr{C}(I, \mathbb{R}^n) \times \mathscr{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}_+ \backslash \{0\}$. *Let* $\bar{u} \in \mathbb{R}^p$. *Then* $e^t \overline{U} f = \bar{u} f$ *and*

$$e^t \mathscr{F}(\overline{U}, f, t, x, \bar{x}, \alpha^1) = \mathscr{F}(\bar{u} f, t, x, \bar{x}, \alpha^1) = \alpha^1(x, \bar{x})\mathscr{F}(\bar{u} f, t, x, \bar{x}, 1).$$

Remark 3.2. Let $f : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$ *and* $g : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^q$ *be continuously differentiable functions. Suppose that there exists differentiable* *function* $y: I \longrightarrow \mathbb{R}^n$*, vector* $u \in \mathbb{R}^p$ *and piecewise smooth function* $u: I \longrightarrow \mathbb{R}^q$ *such that for all* $t \in I$,

$$u f_x(t, y, \dot{y}) + \bar{v}(t) g_x(t, y, \dot{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{u} f_{\dot{x}}(t, y, \dot{y}) + \bar{v}(t) g_{\dot{x}}(t, y, \dot{y})\right),$$

then $\int_a^b \mathscr{F}(uf + v(t)g, t, y, \dot{y}, \mathbf{1})\mathrm{d}t = 0.$

Based upon the concept of the sublinear functional, we suggest the following definitions:

DEFINITION 3.2. (f, h) is said to be (F, α, ρ, d)-type I at $\bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$ if for all $x \in A$ we have

$$\begin{aligned}
&\int_a^b f(t, x, \dot{x})\,\mathrm{d}t - \int_a^b f(t, \bar{x}, \dot{\bar{x}})\,\mathrm{d}t \\
&\quad \geqq \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\,\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\,\mathrm{d}t - \int_a^b h(t, \bar{x}, \dot{\bar{x}})\,\mathrm{d}t \\
&\quad \geqq \int_a^b \mathscr{F}(h, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t.
\end{aligned}$$

DEFINITION 3.3. (f, h) is said to be pseudoquasi (F, α, ρ, d)-type I at $\bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$ if for all $x \in A$ we have

$$\begin{aligned}
&\int_a^b f(t, x, \dot{x})\,\mathrm{d}t < \int_a^b f(t, \bar{x}, \dot{\bar{x}})\,\mathrm{d}t \\
&\quad \Longrightarrow \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t < 0 \qquad \text{(7a)}\\
&- \int_a^b h(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \leqq 0 \\
&\quad \Longrightarrow \int_a^b \mathscr{F}(h, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t \leqq 0. \qquad \text{(7b)}
\end{aligned}$$

If in the above definition, $x \neq \bar{x}$ and inequality (7a) is satisfied as

$$\begin{aligned}
&\int_a^b f(t, x, \dot{x})\mathrm{d}t \leqq \int_a^b f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \\
&\quad \Longrightarrow \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t < 0
\end{aligned}$$

then we say that (f, h) is strictly pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$.

DEFINITION 3.4. (f, h) is said to be weak strictly-pseudoquasi (F, α, ρ, d)-type I at $\bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$ if for all $x \in A$ we have

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t < \int_a^b f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t < 0$$
$$-\int_a^b h(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \leqq 0 \Longrightarrow \int_a^b \mathscr{F}(h, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t \leqq 0.$$

DEFINITION 3.5. (f, h) is said to be strong pseudoquasi (F, α, ρ, d)-type I at $\bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$ if for all $x \in A$ we have

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t \leq \int_a^b f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t \leqslant 0 \tag{8a}$$
$$-\int_a^b h(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \leqq 0$$
$$\Longrightarrow \int_a^b \mathscr{F}(h, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t \leqq 0. \tag{8b}$$

If in the above definition, inequality (8a) is satisfied as

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t < \int_a^b f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t \leqslant 0$$

then we say that (f, h) is weak pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$.

Remark 3.3. Note that for the scalar objective functions the class of pseudoquasi (F, α, ρ, d)-type I, the class of weak strictly-pseudoquasi (F, α, ρ, d)-type I, and the class of strong pseudoquasi (F, α, ρ, d)-type I functions coincide.

DEFINITION 3.6. (f, h) is said to be sub-strictly-pseudoquasi (F, α, ρ, d)-type I at $\bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$ if for all $x \in A$ we have

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqq \int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqslant 0$$
$$-\int_a^b h(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \leqq 0 \Longrightarrow \int_a^b \mathscr{F}(h,t,x,\bar{x},\alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0.$$

DEFINITION 3.7. (f,h) is said to be quasistrictly-pseudo (F,α,ρ,d)-type I at $\bar{x}\in\mathscr{C}(I,\mathbb{R}^n)$ if for all $x\in A$ we have

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqq \int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0 \tag{9a}$$
$$-\int_a^b h(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \leqq 0$$
$$\Longrightarrow \int_a^b \mathscr{F}(h,t,x,\bar{x},\alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqslant 0. \tag{9b}$$

If in the above definition, inequality (9a) is satisfied as

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqslant \int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0$$

then we say that (f,h) is weak quasistrictly-pseudo (F,α,ρ,d)-type I at $\bar{x}$.

DEFINITION 3.8. (f,h) is said to be weak quasisemi-pseudo (F,α,ρ,d)-type I at $\bar{x}\in\mathscr{C}(I,\mathbb{R}^n)$ if for all $x\in A$ we have

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqslant \int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0$$
$$-\int_a^b h(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \leqq 0 \Longrightarrow \int_a^b \mathscr{F}(h,t,x,\bar{x},\alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqslant 0.$$

DEFINITION 3.9. (f, h) is said to be weak strictly-pseudo (F, α, ρ, d)-type I at $\bar{x} \in \mathscr{C}(I, \mathbb{R}^n)$ if for all $x \in A$ we have

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t \leqslant \int_a^b f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t$$
$$\Longrightarrow \int_a^b \mathscr{F}(f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t < 0$$
$$-\int_a^b h(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \leqq 0 \Longrightarrow \int_a^b \mathscr{F}(h, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t < 0.$$

4. Sufficient optimality conditions

In this section, we establish several sufficient optimality conditions for a feasible solution to be efficient, weak efficient or properly efficient for problem (MOP) in the form of the following theorems.

THEOREM 4.1. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vector $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v} : I \longrightarrow \mathbb{R}^q$ such that for all $t \in I$,*

$$\bar{u} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{v}(t) g_x(t, \bar{x}, \dot{\bar{x}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{u} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{v}(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right), \tag{10a}$$
$$\bar{v}(t) g(t, \bar{x}, \dot{\bar{x}}) = 0, \tag{10b}$$
$$\bar{u} > 0, \quad \bar{v}(t) \geqq 0. \tag{10c}$$

Further, if any of the following holds:

(a) $(f(t, ., .), \bar{v}(t)g(t, ., .))$ *is* (F, α, ρ, d)*-type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$*;*
(b) $(\overline{U} f(t, ., .), \bar{v}(t)g(t, ., .))$ *is* (F, α, ρ, d)*-type I at* $\bar{x}$ *with* $e^t\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$*;*
(c) $(\bar{u} f(t, ., .), \bar{v}(t)g(t, ., .))$ *is* (F, α, ρ, d)*-type I at* $\bar{x}$ *with* $\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$*;*

then $\bar{x}$ is a properly efficient solution for (MOP).

Proof.(a) Since $(f(t, ., .), \bar{v}(t)g(t, ., .))$ is (F, α, ρ, d)-type I at $\bar{x}$ and $\bar{u} > 0$, it is easy to see that $(\bar{u} f(t, ., .), \bar{v}(t)g(t, ., .))$ is $(F, \alpha, \rho^\star, d)$-type I at $\bar{x}$ with $\rho^\star = (\bar{u}\rho^1, \rho^2)$. So, the proof of this part follows similar lines as part (c).

(b) Because $(\overline{U} f(t, ., .), \bar{v}(t)g(t, ., .))$ is (F, α, ρ, d)-type I at $\bar{x}$, therefore, for all $x \in A$, we get

$$\int_a^b \overline{U} f(t,x,\dot{x})\mathrm{d}t - \int_a^b \overline{U} f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$

$$\geqq \int_a^b \mathscr{F}(\overline{U} f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{11a}$$

$$0 \geqq \int_a^b \mathscr{F}(\bar{v}(t)g, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{11b}$$

Multiplying (11a) with e^t and using Remark 2.1, we get

$$\int_a^b \bar{u} f(t,x,\dot{x})\mathrm{d}t - \int_a^b \bar{u} f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$

$$\geqq \int_a^b \mathscr{F}(\bar{u} f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + e^t \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{12a}$$

$$0 \geqq \int_a^b \mathscr{F}(\bar{v}(t)g, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{12b}$$

which implies that $(\overline{u} f(t,.,.), \bar{v}(t)g(t,.,.))$ is (F,α,ρ^*,d)-type I at $\bar{x}$ with $\rho^* = (e^t\rho^1, \rho^2)$. Also, the proof of this part follows similar lines as part (c).

(c) Since $(\overline{u} f(t,.,.), \bar{v}(t)g(t,.,.))$ is (F,α,ρ,d)-type I at $\bar{x}$, for any $x \in A$, we have

$$\int_a^b \bar{u} f(t,x,\dot{x})\mathrm{d}t - \int_a^b \bar{u} f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$

$$\geqq \int_a^b \mathscr{F}(\bar{u} f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{13a}$$

$$0 \geqq \int_a^b \mathscr{F}(\bar{v}(t)g, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{13b}$$

Multiplying (13a) and (13b) with $\alpha^1(x,\bar{x})^{-1}$ and $\alpha^2(x,\bar{x})^{-1}$, respectively, we summarize to get

$$\alpha^1(x,\bar{x})^{-1}\left[\int_a^b \bar{u} f(t,x,\dot{x})\mathrm{d}t - \int_a^b \bar{u} f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t\right] \geqq \int_a^b \mathscr{F}(\bar{u} f, t, x, \bar{x}, \mathbf{1})\mathrm{d}t$$
$$+ \int_a^b \mathscr{F}(\bar{v}(t)g, t, x, \bar{x}, \mathbf{1})\mathrm{d}t + (\rho^1\alpha^1(x,\bar{x})^{-1} + \rho^2\alpha^2(x,\bar{x})^{-1})\int_a^b d^2(t,x,\bar{x})\mathrm{d}t$$
$$\geqq \int_a^b \mathscr{F}(\bar{u} f + \bar{v}(t)g, t, x, \bar{x}, \mathbf{1})\mathrm{d}t$$
$$+(\rho^1\alpha^1(x,\bar{x})^{-1} + \rho^2\alpha^2(x,\bar{x})^{-1}) \int_a^b d^2(t,x,\bar{x})\mathrm{d}t$$
$$\geqq (\rho^1\alpha^1(x,\bar{x})^{-1} + \rho^2\alpha^2(x,\bar{x})^{-1}) \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{14}$$

where the above inequalities hold on account of Remark 3.2.

Because $\alpha^1(x,\bar{x})^{-1} > 0$ for all $x \in A$ hence (14) gives

$$\int_a^b \bar{u} f(t,x,\dot{x})\mathrm{d}t \geqq \int_a^b \bar{u} f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t, \quad \text{for all } x \in A \tag{15}$$

which implies $\bar{x}$ minimizes $\int_a^b \bar{u} f(t,x,\dot{x})\mathrm{d}t$ over A with $\bar{u} > 0$. Hence, $\bar{x}$ is a properly efficient solution for (MOP) due to Theorem 1 of Bector and Husain [4]. □

In the following theorem, we prove that the assumptions of the above theorem can be extended to include problems for which $(\bar{u} f(t,.,.), \bar{v}(t)g(t,.,.))$ is pseudoquasi (F,α,ρ,d)-type I function.

THEOREM 4.2. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vector $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v}: I \longrightarrow \mathbb{R}^q$ such that for all $t \in I, (\bar{x},\bar{u},\bar{v})$ satisfies conditions (10) of Theorem 3.1. Further, if $(\bar{u} f(t,.,.), \bar{v}(t)g(t,.,.))$ is pseudoquasi (F,α,ρ,d)-type I at $\bar{x}$ with $\rho^1\alpha^1(\cdot,\bar{x})^{-1} + \rho^2\alpha^2(\cdot,\bar{x})^{-1} \geqq 0$, then $\bar{x}$ is a proper efficient solution for (MOP).*

Proof. From (10b), we have

$$\int_a^b \bar{v}(t)g(t,\bar{x},\dot{\bar{x}})\mathrm{d}t = 0.$$

Because $(\bar{u} f(t,.,.), \bar{v}(t)g(t,.,.))$ is pseudoquasi (F,α,ρ,d)-type I at $\bar{x}$, for any $x \in A$, we get

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\alpha^2)\mathrm{d}t \leqq -\rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t. \tag{16}$$

Since $\alpha^2(x,\bar{x}) > 0$, (16) implies

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\mathbf{1})\mathrm{d}t \leqq -\alpha^2(x,\bar{x})^{-1}\rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t. \tag{17}$$

Equation (17) along with (10a) gives

$$\int_a^b \mathscr{F}(\bar{u} f,t,x,\bar{x},\mathbf{1})\mathrm{d}t \geqq \alpha^2(x,\bar{x})^{-1}\rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t. \tag{18}$$

hence

$$\int_a^b \mathscr{F}(\bar{u} f,t,x,\bar{x},\mathbf{1})\mathrm{d}t + \alpha^1(x,\bar{x})^{-1}\rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t$$
$$\geqq (\alpha^1(x,\bar{x})^{-1}\rho^1 + \alpha^2(x,\bar{x})^{-1}\rho^2) \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \geqq 0 \tag{19}$$

Multiplying (19) with $\alpha^1(x,\bar{x})$, we obtain

$$\int_a^b \mathscr{F}(\bar{u}f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t, x, \bar{x})\mathrm{d}t \geqq 0. \tag{20}$$

Equation (20) along with the fact that $(\bar{u}f(t,.,.), \bar{v}(t)g(t,.,.))$ is pseudoquasi (F,α,ρ,d)-type I at $\bar{x}$, gives (15). Hence, $\bar{x}$ is a properly efficient solution for (MOP). □

About the sufficient conditions for a point to be efficient solution for problem (MOP), we have the following two theorems.

THEOREM 4.3. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vector $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v}: I \longrightarrow \mathbb{R}^p$ such that for all $t \in I$, $(\bar{x}, \bar{u}, \bar{v})$ satisfies conditions (10) of Theorem 3.1. Further, if any of the following holds:*

(a) $(f(t,.,.), \bar{v}(t)g(t,.,.))$ *is strong pseudoquasi* (F,α,ρ,d)*-type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot,\bar{x})^{-1} + \rho^2\alpha^2(\cdot,\bar{x})^{-1} \geqq 0$;
(b) $(\overline{U}f(t,.,.), \bar{v}(t)g(t,.,.))$ *is strong pseudoquasi* (F,α,ρ,d)*- type I at* $\bar{x}$ *with* $e^t\rho^1\alpha^1(\cdot,\bar{x})^{-1} + \rho^2\alpha^2(\cdot,\bar{x})^{-1} \geqq 0$;
(c) $(\bar{u}f(t,.,.), \bar{v}(t)g(t,.,.))$ *is pseudoquasi* (F,α,ρ,d)*-type I at* $\bar{x}$ *with* $\rho^1\alpha^1(\cdot,\bar{x})^{-1} + \rho^2\alpha^2(\cdot,\bar{x})^{-1} \geqq 0$,

then $\bar{x}$ is an efficient solution for (MOP).

Proof. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then there exists $x \in A$ such that

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t \leqslant \int_a^b f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \tag{21a}$$

Multiplying (21a) with $\overline{U}$ and $\bar{u}$, respectively, we get

$$\int_a^b \overline{U}f(t, x, \dot{x})\mathrm{d}t \leqslant \int_a^b \overline{U}f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \tag{21b}$$

$$\int_a^b \bar{u}f(t, x, \dot{x})\mathrm{d}t < \int_a^b \bar{u}f(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t \tag{21c}$$

Since $\bar{v}(t)g(t,\bar{x},\dot{\bar{x}}) = 0$ for all $t \in I$, we have

$$\int_a^b \bar{v}(t)g(t, \bar{x}, \dot{\bar{x}})\mathrm{d}t = 0 \tag{21d}$$

By hypothesis (a) and using (21a) and (21d), we have

$$\int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t+\rho^1\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqslant 0 \tag{22a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\alpha^2)\mathrm{d}t+\rho^2\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqq 0. \tag{22b}$$

Multiplying (22a) and (22b) with $\bar{u}\alpha^1(x,\bar{x})^{-1}$, and $\alpha^2(x,\bar{x})^{-1}$, respectively, we get

$$\int_a^b \mathscr{F}(\bar{u}f,t,x,\bar{x},\mathbf{1})dt<-\bar{u}\alpha^1(x,\bar{x})^{-1}\rho^1\int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{23a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\mathbf{1})dt\leqq\alpha^2(x,\bar{x})^{-1}\rho^2\int_a^b d^2(t,x,\bar{x})\mathrm{d}t. \tag{23b}$$

By the sublinearity of F, we summarize to get

$$\begin{aligned}&\int_a^b \mathscr{F}(\bar{u}f+\bar{v}(t)g,t,x,\bar{x},\mathbf{1})\mathrm{d}t\leqq\int_a^b \mathscr{F}(\bar{u}f,t,x,\bar{x},\mathbf{1})\mathrm{d}t\\&\quad+\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\mathbf{1})\mathrm{d}t<-\bar{u}\rho^1\alpha^1(x,\bar{x})^{-1}\\&\quad+\rho^2\alpha^2(x,\bar{x})^{-1}\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqq 0,\end{aligned} \tag{24}$$

which contradicts (10a) because $F(t,x,\dot{x},\bar{x},\dot{\bar{x}};0)=0$
By hypothesis (b) and using (21b) and (21d), we have

$$\int_a^b \mathscr{F}(\overline{U}f,t,x,\bar{x},\alpha^1)\mathrm{d}t+\rho^1\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqslant 0 \tag{25a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\alpha^2)\mathrm{d}t+\rho^2\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqq 0. \tag{25b}$$

Multiplying (25a) and (25b) with $\alpha^1(x,\bar{x})^{-1}e^t$ and $\alpha^2(x,\bar{x})^{-1}$, respectively, and using Remark 3.1, we get

$$\int_a^b \mathscr{F}(\bar{u}f, t, x, \bar{x}, \mathbf{1})\mathrm{d}t < -\alpha^1(x,\bar{x})^{-1}e^t\rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \tag{26a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g, t, x, \bar{x}, \mathbf{1})\mathrm{d}t \leqq \alpha^2(x,\bar{x})^{-1}\rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t. \tag{26b}$$

By the sublinearity of F, we summarize to get

$$\int_a^b \mathscr{F}(\bar{u}f + \bar{v}(t)g, t, x, \bar{x}, \mathbf{1})\mathrm{d}t$$
$$< -(e^t\rho^1\alpha^1(x,\bar{x})^{-1} + \rho^2\alpha^2(x,\bar{x})^{-1}) \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0, \tag{27}$$

which again contradicts (10a).

By hypothesis (c) and using (21c) and (21d), we have

$$\int_a^b \mathscr{F}(\bar{u}f, t, x, \bar{x}, \alpha^1)\mathrm{d}t + \rho^1 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t < 0 \tag{28a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g, t, x, \bar{x}, \alpha^2)\mathrm{d}t + \rho^2 \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0. \tag{28b}$$

Multiplying (28a) and (28b) with $\alpha^1(x,\bar{x})^{-1}$ and $\alpha^2(x,\bar{x})^{-1}$, respectively, and adding, we obtain

$$\int_a^b \mathscr{F}(\bar{u}f + \bar{v}(t)g, t, x, \bar{x}, \mathbf{1})\mathrm{d}t$$
$$< -(\rho^1\alpha^1(x,\bar{x})^{-1} + \rho^2\alpha^2(x,\bar{x})^{-1}) \int_a^b d^2(t,x,\bar{x})\mathrm{d}t \leqq 0, \tag{29}$$

which again contradicts (10a). □

An interesting case not covered by Theorem 4.3 above is the case where $(\bar{x}, \bar{u}, \bar{v})$ is a solution of (10) but the requirement that $\bar{u} > 0$ is not made. This is given by the following two theorems, where instead of requiring that $\bar{u} > 0$, we enforce other convexity conditions on $(f(t,.,.), \bar{v}(t)g(t,.,.))$.

THEOREM 4.4. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vector $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v}: I \longrightarrow \mathbb{R}^q$ such that for all $t \in I$,*

$$\bar{u}f_x(t,\bar{x},\dot{\bar{x}}) + \bar{v}(t)g_x(t,\bar{x},\dot{\bar{x}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{u}f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{v}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\right), \tag{30a}$$

$$\bar{v}(t)g(t,\bar{x},\dot{\bar{x}}) = 0, \tag{30b}$$

$$\bar{u} \geqslant 0, \quad \bar{v}(t) \geqq 0. \tag{30c}$$

Further, if any of the following holds:

(a) $(f(t,.,.), \bar{v}(t)g(t,.,.))$ *is weak strictly-pseudoquasi* (F,α,ρ,d)*-type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$*;*
(b) $(f(t,.,.), \bar{v}(t)g(t,.,.))$ *is weak quasisemi-pseudo* (F,α,ρ,d)*-type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$*;*
(c) $(\overline{U}f(t,.,.), \bar{v}(t)g(t,.,.))$ *is sub-strictly pseudoquasi* (F,α,ρ,d)*- type I at* $\bar{x}$ *with* $e^t\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$*;*
(d) $(\bar{u}f(t,.,.), \bar{v}(t)g(t,.,.))$ *is strictly pseudoquasi* (F,α,ρ,d)*- type I at* $\bar{x}$ *with* $\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$*;*

then $\bar{x}$ *is an efficient solution for* (MOP).

Proof. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then there exists $x\in A$ such that (21a) holds. Multiplying (21a) with $\overline{U}$ and $\bar{u}$, respectively, we get

$$\int_a^b \overline{U}f(t,x,\dot{x})\mathrm{d}t \leqq \int_a^b \overline{U}f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \tag{31a}$$

$$\int_a^b \bar{u}f(t,x,\dot{x})\mathrm{d}t \leqq \int_a^b \bar{u}f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \tag{31b}$$

By hypothesis (a) and using (21a) and (21d), we have

$$\int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t+\rho^1\int_a^b d^2(t,x,\bar{x})\mathrm{d}t<0 \tag{32a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\alpha^2)\mathrm{d}t+\rho^2\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqq 0. \tag{32b}$$

Multiplying (32a) and (32b) with $\bar{u}\alpha^1(x,\bar{x})^{-1}$ and $\alpha^2(x,\bar{x})^{-1}$, respectively, we get (23). By the sublinearity of F, we summarize to get (24), which contradicts (10a).

By hypothesis (b) and using (21a) and (21d), we have

$$\int_a^b \mathscr{F}(f,t,x,\bar{x},\alpha^1)\mathrm{d}t+\rho^1\int_a^b d^2(t,x,\bar{x})\mathrm{d}t\leqq 0 \tag{33a}$$

$$\int_a^b \mathscr{F}(\bar{v}(t)g,t,x,\bar{x},\alpha^2)\mathrm{d}t+\rho^2\int_a^b d^2(t,x,\bar{x})\mathrm{d}t<0. \tag{33b}$$

Multiplying (33a) and (33b) with $\bar{u}\alpha^1(x,\bar{x})^{-1}$ and $\alpha^2(x,\bar{x})^{-1}$, respectively, and adding, we obtain (24) which contradicts (10a).
By hypothesis (c) and using (31a) and (21d), we have (25) and now the proof follows exactly similar lines as that of part (b) of theorem 4.3.
By hypothesis (d) and using (31b) and (21d), we have (28) and now the proof follows exactly similar lines as that of part (c) of theorem 4.3. □

It is obvious that the Theorems 4.3 and 4.4 hold for weak efficient solutions too. However, it is important to know that the convexity assumptions of Theorems 4.3 and 4.4 can be weakened for weak efficient solutions.

THEOREM 4.5. *Suppose that there exists a feasible solution* $\bar{x}$ *for* (*MOP*) *and vector* $\bar{u}\in\mathbb{R}^p$ *and piecewise smooth function* $\bar{v}: I \longrightarrow \mathbb{R}^q$ *such that for all* $t\in I$, $(\bar{x},\bar{u},\bar{v})$ *satisfies conditions* (*10*) *of Theorem* 4.1. *Further, if any of the following holds:*

(a) $(f(t,.,.),\bar{v}(t)g(t,.,.))$ *is weak pseudoquasi* (F,α,ρ,d)-*type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$;
(b) $(\overline{U}f(t,.,.),\bar{v}(t)g(t,.,.))$ *is weak pseudoquasi* (F,α,ρ,d)-*type I at* $\bar{x}$ *with* $e^t\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$;

then $\bar{x}$ *is a weak efficient solution for* (*MOP*).

Proof. Suppose that $\bar{x}$ is not a weak efficient solution for (MOP). Then there exists $x\in A$ such that

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t < \int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \tag{34a}$$

Multiplying (34a) with $\overline{U}$, we get

$$\int_a^b \overline{U}f(t,x,\dot{x})\mathrm{d}t < \int_a^b \overline{U}f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \tag{34b}$$

By hypothesis (a) and using (34a) and (21d), we obtain (22) and the rest follows exactly similar lines as that of part (a) of theorem 4.3.

By hypothesis (b) and using (34b) and (21d), we obtain (25) and the rest follows exactly similar lines as that of part (b) of theorem 4.3. □

THEOREM 4.6. *Suppose that there exists a feasible solution* $\bar{x}$ *for* (*MOP*) *and vector* $\bar{u}\in\mathbb{R}^p$ *and piecewise smooth function* $\bar{v}: I \longrightarrow \mathbb{R}^q$ *such that for all* $t\in I$, $(\bar{x},\bar{u},\bar{v})$ *satisfies conditions* (*30*) *of Theorem* 4.3. *Further, if any of the following holds:*

(a) $(f(t,.,.),\bar{v}(t)g(t,.,.))$ *is pseudoquasi* (F,α,ρ,d)-*type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$;
(b) $(\overline{U}f(t,.,.),\bar{v}(t)g(t,.,.))$ *is strong pseudoquasi* (F,α,ρ,d)-*type I at* $\bar{x}$ *with* $e^t\rho^1\alpha^1(\cdot,\bar{x})^{-1}+\rho^2\alpha^2(\cdot,\bar{x})^{-1}\geqq 0$;

then $\bar{x}$ *is a weak efficient solution for* (*MOP*).

Proof. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then there exists $x \in A$ such that (34a) holds. Multiplying (34a) with $\overline{U}$, we get (21b). By hypothesis (a) and using (34a) and (21d), we obtain (32) and the rest follows exactly similar lines as that of part (a) of theorem 4.4.
By hypothesis (b) and using (21b) and (21d), we obtain (25) and the rest follows exactly similar lines as that of part (b) of theorem 4.3. □

5. Mixed type duality

The aim of this section is to use the concepts of efficiency, weak efficiency and proper efficiency to formulate duality relationships between the multiobjective variational problem (MOP) and the dual multiobjective variational problem (XMOP) defined as [19]:

$$\text{(XMOP)} \quad \text{Maximize} \int_a^b \{f(t,y,\dot{y}) + [v_{J_1}(t)g^{J_1}(t,y,\dot{y})]e\}\, \mathrm{d}t$$

$$\text{subject to } y(a)=\alpha, \quad y(b)=\beta, \tag{35a}$$

$$uf_x(t,y,\dot{y}) + v(t)g_x(t,y,\dot{y}) = \frac{\mathrm{d}}{\mathrm{d}t}(uf_{\dot{x}}(t,y,\dot{y}) + v(t)g_{\dot{x}}(t,y,\dot{y})), \quad t \in I, \tag{35b}$$

$$v_{J_2}(t)g^{J_2}(t,y,\dot{y}) \geqq 0; \quad t \in I, \tag{35c}$$

$$v(t) \geqq 0; \quad t \in I, \tag{35d}$$

$$u \geqq 0, \quad u^t e = 1. \tag{35e}$$

We note that we get a Mond-Weir dual for $J_1 = \varnothing$ and a Wolfe dual for $J_2 = \varnothing$ in (XMOP), respectively.

5.1. DUALITY AND PROPER EFFICIENCY

Now, we use the concept of proper efficiency to formulate duality relationships between (MOP) and (XMOP). Before proceeding to establish duality results, we state, in the form of the following proposition, the continuous version of Theorem 4.1 of [9], which will be needed in the proof of the strong duality theorem.

PROPOSITION 5.1. *Let $\bar{x}$ be a properly efficient for (MOP) at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v}: I \longrightarrow \mathbb{R}^q$ such that for all $t \in I$, $(\bar{x}, \bar{u}, \bar{v})$ satisfies*

$$\bar{u}f_x(t,\bar{x},\dot{\bar{x}}) + \bar{v}(t)g_x(t,\bar{x},\dot{\bar{x}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{u}f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{v}(t)g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\right), \tag{36a}$$

$$\bar{v}(t)g(t,\bar{x},\dot{\bar{x}}) = 0, \tag{36b}$$

$$\bar{u} > 0; \quad \bar{v}(t) \geqq 0, \tag{36c}$$

$$\bar{u}^t e = 1. \tag{36d}$$

THEOREM 5.1. (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y, u, v) for $(XMOP)$, any of the following holds*:

(a) $(f(t,.,.) + v_{J_1} g^{J_1}(t,.,.)e, v_{J_2} g^{J_2}(t,.,.))$ *is* (F, α, ρ, d)*-type I at y with* $\bar{u}\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$;

(b) $(uf(t,.,.) + v_{J_1} g^{J_1}(t,.,.), v_{J_2} g^{J_2}(t,.,.))$ *is* (F, α, ρ, d)*- type I at y with* $\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(., y)^{-1} \geqq 0$.

Then the following hold

$$\int_a^b \{uf(t, y, \dot{y}) + v_{J_1}(t) g^{J_1}(t, y, \dot{y})\} \mathrm{d}t \leqq \int_a^b uf(t, x, \dot{x}) \mathrm{d}t \tag{37}$$

Proof. (a) Since $(f(t,.,.) + v_{J_1} g^{J_1}(t,.,.)e, v_{J_2} g^{J_2}(t,.,.))$ is (F, α, ρ, d)-type I at $\bar{x}, u \geqq 0$ and $ue = 1$, then $(uf(t,.,.) + v_{J_1} g^{J_1}(t,.,.))e, v_{J_2} g^{J_2}(t,.,.))$ is $(F, \alpha, \rho^\star, d)$-type I at $\bar{x}$ with $\rho^\star = (\bar{u}\rho^1, \rho^2)$. So, the proof of this part follows similar lines as part (b).

(b) Since F is a sublinear functional, from equation (35b) and (35c), we have

$$\int_a^b \mathscr{F}(uf + v(t)g, t, x, y, \mathbf{1}) \, \mathrm{d}t = 0 \tag{38a}$$

$$-\int_a^b v_{J_2}(t) g^{J_2}(t, y, \dot{y}) \, \mathrm{d}t \leqq 0 \tag{38b}$$

Using again the sublinearity of F together with (38a), we obtain

$$\int_a^b \mathscr{F}(uf + v_{J_1}(t) g^{J_1}, t, x, y, \mathbf{1}) \mathrm{d}t \geqq -\int_a^b \mathscr{F}(v_{J_2}(t) g^{J_2}(t, x, y, \mathbf{1}) \mathrm{d}t \tag{38c}$$

Since $(uf(t,.,.) + v_{J_1} g^{J_1}(t,.,.), v_{J_2} g^{J_2}(t,.,.))$ is (F, α, ρ, d)-type I at y, (38b) implies

$$\int_a^b \mathscr{F}(v_{J_2}(t) g^{J_2}, t, x, y, \mathbf{1}) \mathrm{d}t + \alpha^2(x, y)^{-1}\rho^2 \int_a^b d(t, x, y,) \mathrm{d}t \leqq 0 \tag{39}$$

Because $\alpha^1(x, y)^{-1}\rho^1 + \alpha^2(x, y)^{-1}\rho^2 \geqq 0$, inequality (38c) with (39) implies

$$\int_a^b \mathscr{F}(uf + v_{J_1}(t) g^{J_1}, t, x, y, \mathbf{1}) \mathrm{d}t + \alpha^1(x, y)^{-1}\rho^2 \int_a^b d(t, x, y,) \mathrm{d}t \geqq 0. \tag{40}$$

Because $(uf(t,.,.), v_{J_1}g^{J_1}(t,.,.), v_{J_2}g^{J_2}(t,.,.))$ is (F,α,ρ,d)-type I at y, inequality (40) implies

$$\int_a^b \{uf(t,x,\dot{x})+v_{J_1}(t)g^{J_1}(t,x,\dot{x})\}\mathrm{d}t \geqq \int_a^b \{uf(t,y,\dot{y})+v_{J_1}(t)g^{J_1}(t,y,\dot{y})\}\mathrm{d}t \tag{41}$$

Since $x\in A$ and $v(t)\geqq 0$ for all $t\in I$, we get

$$\int_a^b v_{J_1}(t)g^{J_1}(t,x,\dot{x})\mathrm{d}t \leqq 0. \tag{42}$$

Now form inequalities (41) and (42), we obtain

$$\int_a^b \{uf(t,y,\dot{y})+v_{J_1}(t)g^{J_1}(t,y,\dot{y})\}\mathrm{d}t \leqq \int_a^b uf(t,x,\dot{x})\mathrm{d}t.$$ □

THEOREM 5.2. (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y,u,v) for (XMOP), $(uf(t,.,.)+v_{J_1}g^{J_1}(t,.,.), v_{J_2}g^{J_2}(t,.,.))$ is pseudoquasi (F,α,ρ,d)- type I at y with $\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$. Then inequality (37) hold.*

Proof. The proof is similar to that of Theorem 5.1; this can be seen by replacing (F,α,ρ,d)- type I by pseudoquasi (F,α,ρ,d)-type I in the above proof. □

COROLLARY 5.1. *Let $(\bar{y},\bar{u},\bar{v})$ be a feasible solution for (XMOP) with $\bar{u}>0$. Assume that $\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}})=0, t\in I$, and assume that $\bar{y}$ is feasible for (MOP). If weak duality (any of Theorem 5.1 or 5.2) holds between (MOP) and (XMOP). Then, $\bar{y}$ is a properly efficient solution for (MOP) and $(\bar{y},\bar{u},\bar{v})$ is a properly efficient solution for (XMOP).*

Proof. Proceeding on the lines similar to that of [4, Lemma 1]. □

THEOREM 5.3. (Strong Duality). *Let $\bar{x}$ be a properly efficient solution for (MOP) at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist $\bar{u}\in\mathbb{R}^p$ and a piecewise smooth $\bar{v}:I\longrightarrow\mathbb{R}^q$ such that $(\bar{x},\bar{u},\bar{v})$ is feasible for (XMOP), along with the conditions $\bar{v}_{J_1}(t)g^{J_1}(t,\bar{x},\dot{\bar{x}})=0, t\in I$, and $\bar{u}>0$. If also weak duality (Theorem 5.1 or 5.2) holds between (MOP) and (XMOP), then $(\bar{x},\bar{u},\bar{v})$ is a properly efficient solution for (XMOP).*

Proof. Since $\bar{x}$ is a properly efficient solution of (MOP) at which the Kuhn-Tucker constraint qualification is satisfied, by Proposition 5.1, there exist $\bar{u}\in\mathbb{R}^p$ and a piecewise smooth $\bar{v}:I\longrightarrow\mathbb{R}^q$ such that $(\bar{x},\bar{u},\bar{v})$ satisfies (36). Thus $(\bar{x},\bar{u},\bar{v})$ is feasible for (XMOP) with $\bar{u}>0$. Proper efficiency of $(\bar{x},\bar{u},\bar{v})$ for (XMOP) now follows from Corollary 5.1. □

5.2. DUALITY AND EFFICIENCY

Here, we use the concept of efficiency to formulate duality relationships between (MOP) and (XMOP). Before proceeding to establish duality results, we state, in the form of the following proposition, the continuous version of Theorem 3.2 of [23], which will be needed in the proof of the strong duality theorem.

PROPOSITION 5.2. *Let $\bar{x}$ be (weak) efficient for (MOP) at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v}: I \longrightarrow \mathbb{R}^q$ such that, for all $t \in I$, $(\bar{x}, \bar{u}, \bar{v})$ satisfies*

$$\bar{u} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{v}(t) g_x(t, \bar{x}, \dot{\bar{x}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{u} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{v}(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right), \tag{43a}$$

$$\bar{v}(t) g(t, \bar{x}, \dot{\bar{x}}) = 0, \tag{43b}$$

$$\bar{u} \geqq 0, \quad \bar{v}(t) \geqq 0, \tag{43c}$$

$$\bar{u}^t e = 1. \tag{43d}$$

THEOREM 5.4. (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y, u, v) for (XMOP), any of the following holds:*

(a) $u > 0$, *and* $(f(t, ., .) + v_{J_1} g^{J_1}(t, ., .)e, v_{J_2} g^{J_2}(t, ., .))$ *is strong pseudoquasi* (F, α, ρ, d)*-type I at* y *with* $u\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$;
(b) $u > 0$, *and* $(uf(t, ., .) + v_{J_1} g^{J_1}(t, ., .)e, v_{J_2} g^{J_2}(t, ., .))$ *is pseudoquasi* (F, α, ρ, d)*-type I at* y *with* $\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$.

Then the following cannot hold

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t \leqslant \int_a^b \{f(t, y, \dot{y}) + [v_{J_1}(t) g^{J_1}(t, y, \dot{y})]e\} \, \mathrm{d}t \tag{44}$$

Proof. Suppose contrary to the result of the theorem that (44) holds. Since x is feasible for (MOP) and $v(t) \geqq 0$ for all $t \in I$, (44) implies

$$\int_a^b \{f(t, x, \dot{x}) + [v_{J_1}(t) g^{J_1}(t, x, \dot{x})]e\}\mathrm{d}t \leqslant \int_a^b \{f(t, y, \dot{y}) + [v_{J_1}(t) g^{J_1}(t, y, \dot{y})]e\}\mathrm{d}t \tag{45a}$$

Multiplying (45a) with $u > 0$, we get

$$\int_a^b \{uf(t, x, \dot{x}) + [v_{J_1}(t) g^{J_1}(t, x, \dot{x})]\}\mathrm{d}t < \int_a^b \{uf(t, y, \dot{y}) + [v_{J_1}(t) g^{J_1}(t, y, \dot{y})]\}\mathrm{d}t \tag{45b}$$

Since (y, u, v) is feasible for (XMOP), it follows that

$$-\int_a^b v_{J_2}(t) g^{J_2}(t, y, \dot{y}) \mathrm{d}t \leqq 0 \tag{45c}$$

By hypothesis (a), using (45a) and (45c),we have

$$\int_a^b \mathscr{F}(f(t, ., .) + v_{J_1}(t) g^{J_1}(t, ., .)e, t, x, y, \alpha^1) \mathrm{d}t \leqslant -\rho^1 \int_a^b d^2(t, x, y) \mathrm{d}t \tag{46a}$$

$$\int_a^b \mathscr{F}(v_{J_2}(t) g^{J_2} t, x, y, \alpha^2) \mathrm{d}t \leqq -\rho^2 \int_a^b d^2(t, x, y) \mathrm{d}t. \tag{46b}$$

Multiplying (46a) and (46b) with $u\alpha^1(x, y)^{-1}$ and $\alpha^2(x, y)^{-1}$, respectively, by sublinearity of F, we obtain

$$\begin{aligned}\int_a^b \mathscr{F}(uf + v(t)g, t, x, y, \mathbf{1}) \mathrm{d}t &\leqq \int_a^b \mathscr{F}(uf + v_{J_1}(t) g^{J_1}, t, x, y, \mathbf{1}) \mathrm{d}t \\ &\quad + \int_a^b \mathscr{F}(v_{J_2}(t) g^{J_2}, t, x, y, \mathbf{1}) \mathrm{d}t \\ &< -(u\rho^1 \alpha^1(x, y)^{-1} \\ &\quad + \rho^2 \alpha^2(x, y)^{-1}) \int_a^b d^2(t, x, y) \mathrm{d}t \end{aligned} \tag{47}$$

which contradicts the duality constraint (35b) on account of Remark 3.2. Hence, (44) cannot hold.
When hypothesis (b) holds, inequalities (45b) and (45c) implies

$$\int_a^b \mathscr{F}(uf + v_{J_1}(t) g^{J_1}, t, x, y, \alpha^1) \mathrm{d}t < -\rho^1 \int_a^b d^2(t, x, y) \mathrm{d}t \tag{48a}$$

$$\int_a^b \mathscr{F}(v_{J_2}(t) g^{J_2}, t, x, y, \alpha^2) \mathrm{d}t \leqq -\rho^2 \int_a^b d^2(t, x, y) \mathrm{d}t. \tag{48b}$$

Multiplying (48a) and (48b) with $\alpha^1(x, y)^{-1}$ and $\alpha^2(x, y)^{-1}$, respectively, by sublinearity of F, we obtain

$$\begin{aligned}&\int_a^b \mathscr{F}(uf + v(t)g, t, x, y, \mathbf{1}) \mathrm{d}t < -(\rho^1 \alpha^1(x, y)^{-1} \\ &\quad + \rho^2 \alpha^2(x, y)^{-1}) \int_a^b d^2(t, x, y) \mathrm{d}t \end{aligned} \tag{49}$$

which again contradicts the duality constraint (35b). □

THEOREM 5.5. (Weak duality). *Assume that for all feasible x for (MOP) and all feasible (y,u,v) for $(XMOP)$, any of the following holds:*

(a) $(f(t,.,.)+v_{J_1}g^{J_1}(t,.,.)e, v_{J_2}g^{J_2}(t,.,.))$ *is weak strictly-pseudoquasi* (F,α,ρ,d)*-type I at y with* $u\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$;
(b) $(uf(t,.,.)+v_{J_1}g^{J_1}(t,.,.), v_{J_2}g^{J_2}(t,.,.))$ *is strictly pseudoquasi* (F,α,ρ,d)*-type I at y with* $\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$;
(c) $(f(t,.,.)+v_{J_1}g^{J_1}(t,.,.)e, v_{J_2}g^{J_2}(t,.,.))$ *is weak quasistrictly-pseudo* (F,α,ρ,d)*-type I at y with* $u\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$;
(d) $(uf(t,.,.)+v_{J_1}g^{J_1}(t,.,.), v_{J_2}g^{J_2}(t,.,.))$ *is quasistrictly-pseudo* (F,α,ρ,d)*-type I at y with* $\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$.

Then inequality (44) cannot hold.

Proof. Suppose contrary to the result of the theorem that (44) holds. Since x is feasible for (MOP) and $v(t)\geqq 0$ for all $t\in I$, (44) implies (45a). Since (y,u,v) is feasible for (XMOP), we have (45c). Multiplying (45a) with $u\geqslant 0$, we get

$$\int_a^b \{uf(t,x,\dot{x})+[v_{J_1}(t)g^{J_1}(t,x,\dot{x})]\}\,\mathrm{d}t \leqq \int_a^b \{uf(t,y,\dot{y}) +[v_{J_1}(t)g^{J_1}(t,y,\dot{y})]\}\,\mathrm{d}t \tag{50}$$

Now if hypothesis (a) holds, using (45a) and (45c), we get

$$\int_a^b \mathscr{F}(f(t,.,.)+v_{J_1}(t)g^{J_1}(t,.,.)e,t,x,y,\alpha^1)\mathrm{d}t < -\rho^1\int_a^b d^2(t,x,y)\mathrm{d}t \tag{51a}$$

$$\int_a^b \mathscr{F}(v_{J_2}(t)g^{J_2},t,x,y,\alpha^2)\mathrm{d}t \leqq -\rho^2\int_a^b d^2(t,x,y)\mathrm{d}t. \tag{51b}$$

Multiplying (51a) and (51b) with $u\alpha^1(x,y)^{-1}$ and $\alpha^2(x,y)^{-1}$, respectively, using sublinearity of F, we get (47) which contradicts (35b).
Now by hypothesis (b) and, using (50) and (45c), we get (48) and then we obtain (49) again contradicting (35b).

If hypothesis (c) holds, then (45a) and (45c) implies

$$\int_a^b \mathscr{F}(f(t,.,.)+v_{J_1}(t)g^{J_1}(t,.,.)e,t,x,y,\alpha^1)\mathrm{d}t \tag{52a}$$
$$\leqq -\rho^1\int_a^b d^2(t,x,y)\mathrm{d}t$$

$$\int_a^b \mathscr{F}(v_{J_2}(t)g^{J_2},t,x,y,\alpha^2)\mathrm{d}t < -\rho^2\int_a^b d^2(t,x,y)\mathrm{d}t. \tag{52b}$$

Multiplying (52a) and (52b) with $u\alpha^1(x,y)^{-1}$ and $\alpha^2(x,y)^{-1}$, respectively, using sublinearity of F, we get (47) which contradicts (35b).

Now by hypothesis (d), (50) and (45c) leads to

$$\int_a^b \mathscr{F}(uf + v_{J_1}(t)g^{J_1}, t, x, y, \alpha^1)\mathrm{d}t \leqq -\rho^1 \int_a^b d^2(t,x,y)\mathrm{d}t \tag{53a}$$

$$\int_a^b \mathscr{F}(v_{J_2}(t)g^{J_2}, t, x, y, \alpha^2)\mathrm{d}t < -\rho^2 \int_a^b d^2(t,x,y)\mathrm{d}t. \tag{53b}$$

Multiplying (53a) and (53b) with $\alpha^1(x,y)^{-1}$ and $\alpha^2(x,y)^{-1}$, respectively, using sublinearity of F, we get (49) which again contradicts (35b). □

COROLLARY 5.2. *Let* $(\bar{y},\bar{u},\bar{v})$ *be a feasible solution for problem* (*XMOP*). *Assume that* $\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}}) = 0$, *for all* $t \in I$ *and assume that* $\bar{y}$ *is feasible for* (*MOP*). *If weak duality* (*any of Theorem 5.4 or 5.5*) *holds between* (*MOP*) *and* (*XMOP*). *Then,* $\bar{y}$ *is an efficient solution for* (*MOP*) *and* $(\bar{y},\bar{u},\bar{v})$ *is an efficient solution for* (*XMOP*).

Proof. Suppose that $\bar{y}$ is not an efficient solution for (MOP), then there exists feasible x for (MOP) such that

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqslant \int_a^b f(t,\bar{y},\dot{\bar{y}})\mathrm{d}t$$

and since $\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}}) = 0$, for all $t \in [a,b]$, we get

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqslant \int_a^b \{f(t,\bar{y},\dot{\bar{y}}) + [\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}})]e\}\mathrm{d}t$$

Since $(\bar{y},\bar{u},\bar{v})$ is feasible for (XMOP) and x is feasible for (MOP), this inequality contradicts the weak duality (Theorem 5.4 or 5.5).

Also suppose that $(\bar{y},\bar{u},\bar{v})$ is not an efficient solution for (XMOP). Then there exists a feasible solution (y,u,v) for (XMOP) such that

$$\int_a^b \{f(t,\bar{y},\dot{\bar{y}}) + [\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}})]e\}\mathrm{d}t \leqslant \int_a^b \{f(t,y,\dot{y}) + \left[v_{J_1}(t)g^{J_1}(t,y,\dot{y})\right]e\}\mathrm{d}t \tag{54}$$

and since $\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}}) = 0$, for all $t \in [a,b]$, (54) reduced to

$$\int_a^b f(t,\bar{y},\dot{\bar{y}})\mathrm{d}t \leqslant \int_a^b \{f(t,y,\dot{y}) + \left[v_{J_1}(t)g^{J_1}(t,y,\dot{y})\right]e\}\mathrm{d}t \tag{55}$$

Since $\bar{y}$ is feasible for (MOP), this inequality contradicts weak duality (Theorem 5.4 or 5.5). Therefore $\bar{y}$ and $(\bar{y},\bar{u},\bar{v})$ are efficient solutions for their respective programs. □

THEOREM 5.6. (Strong duality). *Let $\bar{x}$ be an efficient solution for (MOP) at which the Khun Tucker qualification constraint is satisfied, then there exist $\bar{u} \in \mathbb{R}^p$ and piecewise smooth function $\bar{v} : I \longrightarrow \mathbb{R}^q$ such that $(\bar{x}, \bar{u}, \bar{v})$ is feasible for ($XMOP$) with $\bar{v}_{J_1}(t) g^{J_1}(t, \bar{x}, \dot{\bar{x}}) = 0, t \in I$. If also weak duality (Theorem 5.4 or 5.5) holds between (MOP) and ($XMOP$), then $(\bar{x}, \bar{u}, \bar{v})$ is an efficient solution for ($XMOP$).*

Proof. Since $\bar{x}$ is an efficient solution for (MOP) at which the Kuhn-Tucker constraint qualification is satisfied, by Proposition 5.2, there exist $\bar{u} \in \mathbb{R}^p$ and a piecewise smooth $\bar{v} : I \longrightarrow \mathbb{R}^q$ such that $(\bar{x}, \bar{u}, \bar{v})$ satisfies (43). Thus $(\bar{x}, \bar{u}, \bar{v})$ is feasible for (XMOP). Efficiency of $(\bar{x}, \bar{u}, \bar{v})$ for (XMOP) now follows from Corollary 5.2. □

THEOREM 5.7. (Strict converse duality). *Let $\bar{x}$ be feasible solution for problem (MOP) and $(\bar{y}, \bar{u}, \bar{v})$ be feasible solution for problem ($XMOP$) such that*

$$\int_a^b \bar{u} f(t, \bar{x}, \dot{\bar{x}}) dt = \int_a^b \{\bar{u} f(t, \bar{y}, \dot{\bar{y}}) + \bar{v}_{J_1}(t) g^{J_1}(t, \bar{y}, \dot{\bar{y}})\} \; dt \tag{56}$$

If condition (b) *or* (d) *of Theorem 4.5 is satisfied for $\bar{x}$ and $(\bar{y}, \bar{u}, \bar{v})$, then $\bar{x} = \bar{y}$.*

Proof. We assume $\bar{x} \neq \bar{y}$ and exhibit a contradiction. Since $\bar{x}$ and $(\bar{y}, \bar{u}, \bar{v})$ are feasible for (MOP) and (XMOP), respectively, then $\bar{v}(t) \geqq 0, g(t, \bar{x}, \dot{\bar{x}}) \leqq 0$ for all $t \in I$ and, (56) and (35c) yield

$$\int_a^b \{\bar{u} f(t, \bar{x}, \dot{\bar{x}}) + \bar{v}_{J_1(t)} g^{J_1}(t, \bar{x}, \dot{\bar{x}})\} \mathrm{d}t \leqq \int_a^b \{\bar{u} f(t, \bar{y}, \dot{\bar{y}}) + \bar{v}_{J_1(t)} g^{J_1}(t, \bar{y}, \dot{\bar{y}})\} \; \mathrm{d}t \tag{57a}$$

$$-\int_a^b \bar{v} J_2(t) g J_2(t, \bar{y}, \dot{\bar{y}}) \mathrm{d}t \leqq 0 \tag{57b}$$

Condition (b) of Theorem 5.5 with (57), gives

$$\int_a^b \mathscr{F}(\bar{u} f + \bar{v}_{J_1}(t) g^{J_1}, t, \bar{x}, \bar{y}, \alpha^1) \mathrm{d}t < -\rho^1 \int_a^b d^2(t, \bar{x}, \bar{y}) \mathrm{d}t \tag{58a}$$

$$\int_a^b \mathscr{F}(\bar{v} J_2(t) g^{J_2}, t, \bar{x}, \bar{y}, \alpha^2) \mathrm{d}t \leqq -\rho^2 \int_a^b d^2(t, \bar{x}, \bar{y}) \mathrm{d}t. \tag{58b}$$

Condition (d) of Theorem 5.5 with (57), gives

$$\int_a^b \mathscr{F}(\bar{u} f + \bar{v}_{J_1}(t) g^{J_1}, t, \bar{x}, \bar{y}, \alpha^1) \mathrm{d}t \leqq -\rho^1 \int_a^b d^2(t, \bar{x}, \bar{y}) \mathrm{d}t \tag{59a}$$

$$\int_a^b \mathscr{F}(\bar{v}_{J_2}(t) g^{J_2}, t, \bar{x}, \bar{y}, \alpha^2) \mathrm{d}t < -\rho^2 \int_a^b d^2(t, \bar{x}, \bar{y}) \mathrm{d}t. \tag{59b}$$

By sublinearity of F, both systems (58) and (59) give

$$\int_a^b \mathscr{F}(\bar{u}f+\bar{v}(t)g,t,\bar{x},\bar{y},\mathbf{1})\mathrm{d}t<-(\rho^1\alpha^1(\bar{x},\bar{y})^{-1}+\rho^2\alpha^2(\bar{x},\bar{y})^{-1})\int_a^b d^2(t,\bar{x},\bar{y})\mathrm{d}t \tag{60}$$

which contradicts the duality constraint (35b). □

THEOREM 5.8. (Strict converse duality). *Let $\bar{x}$ be feasible solution for problem (MOP) and $(\bar{y},\bar{u},\bar{v})$ be feasible solution for problem $(XMOP)$ such that*

$$\int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t=\int_a^b\{f(t,\bar{y},\dot{y})+\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}})e\}\mathrm{d}t \tag{61}$$

For each feasible x for (MOP) and (y,u,v,p) for $(XMOP)$,

(a) *if weak duality (any of Theorem 5.4 or 5.5) holds at $\bar{y}$, then $\bar{x}$ is efficient for (MOP);*
(b) *if weak duality (any of Theorem 5.4 or 5.5) holds at y, then $(\bar{y},\bar{u},\bar{v})$ is efficient for $(XMOP)$.*

Proof. (a) Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then, there exist a feasible x for (MOP) such that

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t\leqslant\int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t$$

using condition (61) we get

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t\leqslant\int_a^b\{f(t,\bar{y},\dot{\bar{y}})+\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}})e\}\mathrm{d}t \tag{62}$$

which contradicts the weak duality for feasible solutions x for (MOP) and $(\bar{y},\bar{u},\bar{v})$ for (XMOP). Thus, $\bar{x}$ is efficient for (MOP).

(b) Let as assume on the contrary that $(\bar{y},\bar{u},\bar{v})$ is not an efficient solution for (XMOP). Then, there exist a feasible (y,u,v) for (XMOP) such that

$$\int_a^b\{f(t,\bar{y},\dot{\bar{y}})+\bar{v}_{J_1}(t)g^{J_1}(t,\bar{y},\dot{\bar{y}})e\}\mathrm{d}t\leqslant\int_a^b\{f(t,y,\dot{y})+v_{J_1}(t)g^{J_1}(t,y,\dot{y})e\}\mathrm{d}t. \tag{63}$$

Using condition (61) we get

$$\int_a^b f(t,\bar{x},\dot{\bar{x}})\mathrm{d}t \leqslant \int_a^b \{f(t,y,\dot{y}) + v_{J_1}(t)g^{J_1}(t,y,\dot{y})e\}\mathrm{d}t \tag{64}$$

which contradicts the weak duality for feasible solutions $\bar{x}$ for (MOP) and (y,u,v) for (XMOP). Thus, $(\bar{y},\bar{u},\bar{v})$ is efficient for (XMOP). □

5.3. DUALITY AND WEAK EFFICIENCY

In this section, we use the concept of weak efficiency to formulate duality relationships between (MOP) and (XMOP).

THEOREM 5.9. (Weak duality). *Assume that for all feasible x for (MOP) and all feasible (y,u,v) for (XMOP), any of the following holds:*

(a) $u>0$, *and* $(f(t,.,.)+v_{J_1}g^{J_1}(t,.,.)e, v_{J_2}g^{J_2}(t,.,.))$ *is weak pseudoquasi* (F,α,ρ,d)*-type I at y with* $u\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$*;*
(b) $(f(t,\cdot,\cdot)+v_{J_1}g^{J_1}(t,.,.)e, v_{J_2}g^{J_2}(t,.,))$ *is pseudoquasi* (F,α,ρ,d)*-type I at y with* $u\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$*;*
(c) $(uf(t,.,.) + v_{J_1}g^{J_1}(t,.,.), v_{J_2}g^{J_2}(t,.,.))$ *is pseudoquasi* (F,α,ρ,d)*-type I at y with* $\rho^1\alpha^1(\cdot,y)^{-1}+\rho^2\alpha^2(\cdot,y)^{-1}\geqq 0$.

Then the following cannot hold

$$\int_a^b f(t,x,\dot{x})\ \mathrm{d}t < \int_a^b \{f(t,y,\dot{y}) + [v_{J_1}(t)g^{J_1}(t,y,\dot{y})]e\}\mathrm{d}t \tag{65}$$

Proof. Suppose contrary to the result of the theorem that (65) holds. Since x is feasible for (MOP) and $v(t)\geqq 0$ for all $t\in I$, (65) implies

$$\int_a^b \{f(t,x,\dot{x}) + [v_{J_1}(t)g^{J_1}(t,x,\dot{x})]e\}\mathrm{d}t < \int_a^b \{f(t,y,\dot{y}) + [v_{J_1}(t)g^{J_1}(t,y,\dot{y})]e\}\mathrm{d}t \tag{66}$$

Multiplying (66) with $u\geqslant 0$, we get

$$\int_a^b \{uf(t,x,\dot{x}) + [v_{J_1}(t)g^{J_1}(t,x,\dot{x})]\}\mathrm{d}t < \int_a^b \{uf(t,y,\dot{y}) + [v_{J_1}(t)g^{J_1}(t,y,\dot{y})]\}\mathrm{d}t \tag{67}$$

Since (y,u,v) is feasible for (XMOP), we have (45c).
Now by hypothesis (a) and, from (66) and (45c) we get (46), then we obtain (47) which contradicts (35b).

Now by hypothesis (b) and, from (66) and (45c) we get (51), then we obtain (47) which contradicts (35b).

Now by hypothesis (c) and, from (67) and (45c) we get (48), then we obtain (49) which contradicts (35b). □

COROLLARY 5.3. *Let* $(\bar{y}, \bar{u}, \bar{v})$ *be a feasible solution for problem* (*XMOP*). *Assume that* $\bar{v}_{J_1}(t)g^{J_1}(t, \bar{y}, \dot{\bar{y}}) = 0, t \in I$, *and assume that* $\bar{y}$ *is feasible for* (*MOP*). *If weak duality* (*Theorem 5.9*) *holds between* (*MOP*) *and* (*XMOP*). *Then,* $\bar{y}$ *is a weak efficient solution for* (*MOP*) *and* $(\bar{y}, \bar{u}, \bar{v})$ *is a weak efficient solution for* (*XMOP*).

Proof. Suppose that $\bar{y}$ is not a weak efficient solution for (MOP), then there exists feasible x for (MOP) such that

$$\int_a^b f(t, x, \dot{x})\mathrm{dt} < \int_a^b f(t, \bar{y}, \dot{\bar{y}})\mathrm{d}t$$

and since $\bar{v}_{J_1}(t)g^{J_1}(t, \bar{y}, \dot{\bar{y}}) = 0$, for all $t \in I$, we get

$$\int_a^b f(t, x, \dot{x})\mathrm{d}t < \int_a^b \{f(t, \bar{y}, \dot{\bar{y}}) + [\bar{v}_{J_1}(t)g^{J_1}(t, \bar{y}, \dot{\bar{y}})]e\}\mathrm{d}t$$

Since $(\bar{y}, \bar{u}, \bar{v})$ is feasible for (XMOP) and x is feasible for (MOP), this inequality contradicts the weak duality (Theorem 5.9).

Also suppose that $(\bar{y}, \bar{u}, \bar{v})$ is not a weak efficient solution for (XMOP). Then there exists a feasible solution (y, u, v) for (XMOP) such that

$$\int_a^b \{f(t, \bar{y}, \dot{\bar{y}}) + [\bar{v}_{J_1}(t)g^{J_1}(t, \bar{y}, \dot{\bar{y}})]e\}\mathrm{d}t < \int_a^b \{f(t, y, \dot{y}) + [v_{J_1}(t)g^{J_1}(t, y, \dot{y})]e\}\mathrm{d}t \tag{68}$$

and since $\bar{v}_{J_1}(t)g^{J_1}(t, \bar{y}, \dot{\bar{y}}) = 0$, for all $t \in I$, (68) reduced to

$$\int_a^b f(t, \bar{y}, \dot{\bar{y}})\mathrm{d}t < \int_a^b \{f(t, y, \dot{y}) + [v_{J_1}(t)g^{J_1}(t, y, \dot{y})]e\}\mathrm{d}t$$

Since $\bar{y}$ is feasible for (MOP), this inequality contradicts weak duality (Theorem 5.9). Therefore $\bar{y}$ and $(\bar{y}, \bar{u}, \bar{v})$ are weak efficient solutions for their respective programs.

THEOREM 5.10. (Strong duality). *Let* $\bar{x}$ *be a weak efficient solution for* (*MOP*) *at which the Khun Tucker qualification constraint is satisfied, then there exist* $\bar{u} \in \mathbb{R}^p$ *and piecewise smooth function* $\bar{v} : I \longrightarrow \mathbb{R}^q$ *such that* $(\bar{x}, \bar{u}, \bar{v})$ *is feasible for* (*XMOP*) *with* $\bar{v}_{J_1}(t)g^{J_1}(t, \bar{x}, \dot{\bar{x}}) = 0, t \in I$. *If also weak duality* (*Theorem 5.9*) *holds between* (*MOP*) *and* (*XMOP*), *then* $(\bar{x}, \bar{u}, \bar{v})$ *is a weak efficient solution for* (*XMOP*).

Proof. Since $\bar{x}$ is a weak efficient solution of (MOP) at which the Kuhn–Tucker constraint qualification is satisfied, by Proposition 5.2, there exist $\bar{u} \in \mathbb{R}^p$ and a piecewise smooth $\bar{v}: I \longrightarrow \mathbb{R}^q$ such that $(\bar{x}, \bar{u}, \bar{v})$ satisfies (43). Thus $(\bar{x}, \bar{u}, \bar{v})$ is feasible for (XMOP). Weak efficiency of $(\bar{x}, \bar{u}, \bar{v})$ for (XMOP) now follows from Corollary 5.3. □

References

1. Aghezzaf, B. and Hachimi, M. (2000), Generalized invexity and duality in multiobjective programming problems, *Journal of Global Optimization* 18, 91–101.
2. Aghezzaf, B. and Hachimi, M. (2001), Sufficiency and duality in multiobjective programming Involving Generalized (F, ρ)-convexity, *Journal of Mathematical Analysis and Application* 258, 617–628.
3. Aghezzaf, B. and Khazafi, K. (2002), Generalized B-invexity and duality in multiobjective variational programming problems, Proceedings of the Fourth International Conference on Applied Mathematics and Engineering Sciences, CIMASI, 23–25 october, Casablanca, Morocco.
4. Bector, C.R. and Husain, I. (1992), Duality for multiobjective variational problems, *Journal of mathenmatical Analysis and Application* 166, 214–229.
5. Bhatia, D. and Mehra, A. (1999), Optimality conditions and duality for multiobjective variational problems with generalized B-invexity, *Journal of mathenmatical Analysis and Application* 234, 341–360.
6. Bhatia, D. and Kumar, P. (1996), Duality for multiobjective variational problems with B-vex functions, optimization 36, 347–360.
7. Bector, C.R. Chandra, S. and Abha, (2001), On mixed duality in mathematical progrmming, *Journal of mathenmatical Analysis and Application* 259, 346–356.
8. Egudo, R. (1989), Efficiency and generalized convex duality for multiobjective programs, *Journal of mathenmatical Analysis and Application* 138, 84–94.
9. Gulati, T.R. and Islam, M.A. (1994), Sufficiency and duality in multiobjective programming involving generalized F-convex functions, *Journal of mathenmatical Analysis and Application* 183, 181–195.
10. Hachimi, M. and Aghezzaf, B. (2004), Sufficiency and duality in differentiable multiobjective programming involving generalized type I functions, *Journal of Mathematical Analysis and Application* 296, 382–392.
11. Hanson, M.A. (1981), On Sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Application* 80, 545–550.
12. Hanson, M.A. and Mond, B. (1982), Further generalizations of convexity in mathematical programming, *Journal of Information Optimization Science* 3 25–32.
13. Kaul, R.N., Suneja, S.K. and Srivastava, M.K. (1994), Optimality criteria and duality in multiobjective optimization involving generalized invexity, *Journal of Optimization Theory and Application* 80 465–482.
14. Liu, J. C. and Wu, C.S. (1998), On minimax fractional optimality conditions with (F, ρ)-convexity, *Journal of Mathematical Analysis and Application* 219, 36–51.
15. Maeda, T. (1994), Constraint qualification in multiobjective optimization problems : Differentiable case, *Jornal of Optimization Theory Application* 80 483–500.
16. Mond, B. and Hanson, M.A. (1967), Duality for variational problems, *Journal of Mathematical Analysis and Application* 18, 355–364.
17. Mond, B., Chandra, S. and Husain, I. (1988), Duality for variational problems with invexity, *Journal of Mathematical Analysis and Application* 134, 322–328.

18. Mishra, S. K. and Mukherjee, R.N. (1994), On efficiency and duality for multiobjective variational problems, *Journal of Mathematical Analysis and Application* 187, 40–54.
19. Mukherjee, R.N. and Purnachandra Rao, Ch. (2000), Mixed type duality for multiobjective variational problems, *Journal of Mathematical Analysis and Application* 252, 571–586.
20. Nahak,C. and Nanda, C. (1996), Duality for multiobjective variational problems with invexity, Optimization 36, 235–248.
21. Preda, V. (1992), On Sufficiency and duality for multiobjective program, *Journal of Mathematical Analysis and Application* 166, 365–377.
22. Rueda, N.G. and Hanson, M.A. (1988), Optimality criteria in mathematical programming involving generalized invexity, *Journal of Mathematical Analysis and Application* 130, 375–385.
23. Weir, T. and Mond, B. (1989), Generalized convexity and duality in multiple objective programming, Bulletin of the Australian Mathematical Society 39, 287–299.
24. Xu, Z. (1996), Mixed type duality in multiobjective programming problems, *Journal of Mathematical Analysis and Application* 198, 621–635.